AN INVESTIGATION OF THE MATHEMATICAL RELATIONS OF ZERO AND INFINITY.

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"WE have", says Pascal, "three principal objects in the study of truth:—One to discover it; another to demonstrate it; and a third to discriminate it from the false, when it is examined".

The object of the present article is to *discriminate the true from the false* with respect to mathematical infinity—the *infinitely great* and the *infinitely small*. We shall endeavor to show that these terms admit of *exact* definition, and consequently of a *strictly logical* calculus.

Let it be premised however that we set up no claim to originality, as we shall employ these terms in the sense in which they have been used by such accurate thinkers as Locke, Sturm, Du Hamel, De Morgan and others: and further, lest it be objected that because the discussion contains little that is new or original, it is therefore useless, we shall call attention to the misuse of these terms in some of our best Text-books, and to the confusion and error arising therefrom.

We begin by asking the reader to admit the following Postulates:—

(1). A number, or other quantity, as a distance, may be conceived as increasing without limit. We mean that we can conceive of no number so great but that we can readily conceive a greater. We cannot conceive of space as limited, beyond which limit there is no space; and we can not conceive a number beyond which there is no number. Let it be granted then that number and space are *infinite*, in the sense of *unlimited*.

(2). Let x be a variable. That is, let x be a symbol which may represent any and every number. Then by (1) x may be increased without limit.

(3). Since x may be increased *without limit*, it may be considered as greater than any *constant* whatever, i. e., greater than any *assignable* num-

ber. The value of x being wholly arbitrary, the instant we assign a value to any other symbol, as a, we may suppose $x > a$.

(4). We employ the symbol ∞ to represent a *variable* which increases indefinitely, and which, by reason of its indefiniteness, may be considered greater than any *assignable* number. The expression, $x = \infty$, may be read "when x increases without limit"; or, "when x increases indefinitely"; or even, "when x is infinite" (is unlimited), not "when x is equal to infinity". As, however, we require a name for this symbol, ∞, we shall call it infinity.

(5). *Infinity*, then, is a *variable*, considered as greater than any *assignable* number, and as increasing *without limit*.

According to this definition *infinity* is not a value that can be "reached", or that can be "approached"; nor is it a *limit* of any value, or values.

In Davies and Peck's Mathematical Dictionary, article Infinity, we find the following very singular statement:

"Mathematically considered, *infinity* is always a limit of a variable quantity". If now we turn to the word Limit, we find that " a limit is a quantity towards which a varying quantity may approach within less than any assignable quantity but which it cannot pass". Did these writers really mean to assert that they had any notion of infinity as a value that might be "approached within less than any assignable quantity"? They appear to reach the notion in this way. Consider the relation

$$t = \frac{a}{m-n}.$$

If a is a constant and n is very nearly equal to m then t is very great; and "finally, when n equals m, t is infinite". Just here is the source of all the confusion and error. The quantity t is not infinite, they hold, so long as $m-n$ differs from zero; but "finally when $m = n$" or when $m - n = 0$, then $t = \infty$. Observe that the symbol 0 is put for $m - m$, it is the absolute nought. If, however, we turn to the article Zero we find that when

$$\frac{a}{\infty} = 0,$$

"this kind of 0 differs analytically from the absolute nought, obtained by subtracting a from a; $a - a = 0$". "*It is in consequence of confounding the 0 arising from dividing a by ∞ with the absolute 0 that so much confusion has been created in the discussions that have grown out of this subject*". (Italics mine.) It appears then, that when we write

$$\frac{a}{0} = \infty \text{ and } \frac{a}{\infty} = 0$$

we have two different kinds of nought. No wonder there is confusion. However, when we turn to the article Nothing, we are consoled by the

statement that "*Nothing* is fast falling into disuse as a mathematical term, and the proper term zero (= infinitesimal) is acquiring its true place in the mathematical vocabulary". Absolute zero is worth nothing; we will therefore throw it away. Are we then, when the mathematical millenium shall have arrived, to write $a - a =$ infinitesimal?

To complete the "confusion", we are told that "In arithmetic *infinity* is the last term of the series of natural numbers". There is then a last term of the series 1, 2, 3, ... which cannot be passed!

Professor Hackly (see Algebra, pp. 175–6) admits that $+\infty$ is the limit of increasing magnitudes and $-\infty$ the limit of decreasing magnitudes; and further that if we have the relation

$$x = \frac{m}{n-z}$$

and z increases from a value less than n to a value equal to n x becomes $+\infty$, while if z decreases from a value greater than n to a value equal to n x becomes equal to $-\infty$. *Minus* infinity therefore differs from *plus* infinity only as *minus* zero differs from *plus* zero.

We think the foregoing references are sufficient to show that the subject demands reinvestigation. We now return from this digression.

(6). If one variable x, may increase without limit, then its reciprocal $1 \div x$, will decrease without limit. If we put $z = 1 \div x$ then as x increases indefinitely z decreases indefinitely. As x never ceases to increase, z never ceases to decrease ; or, in other words, as x never reaches any limit of increase, so z can never reach any limit of decrease.

(7). *A variable which decreases indefinitely and which, by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinites'l. Thus, read when $x = \circ$, when x decreases indefi'ly, or when x is an infinites'l.

(8). If a be a constant, the expressions

$$\frac{a}{\circ} = \infty \text{ and } \frac{a}{\infty} = \circ$$

are rigidly exact. The first asserts that if the numerator of a fract. is const. and the denominator decreases indefinitely the value of the fraction increases without limit. The second asserts that if the denominator increases without limit the value of the fraction decreases indefinitely. Though the value approaches nought as a limit, it decreases *without limit* since it can never become nought by any increase of the denominator.

(9). The *limit* of a variable is a *constant* which the variable indefinitely approaches.

Cor. I. The variable can never reach its limit, otherwise the approach would not be indefinite.

[Though the foregoing Proposition and Corollary are true when referred to the particular variable alluded to, and many others, yet they can not be accepted as generally true. For instance, the limits of a variable sine, or cosine of an arc, are zero and unity, constants, either of which the variable may reach.—Ed.]

Cor. II. Since the symbols ∞ and $\circ$ represent values wholly indeterminate and unassignable, it is evident that we may have the following relat'ns,

$$\infty \pm a = \infty; \ \infty \times a = \infty; \ \infty \div a = \infty.$$

The symbols ∞ in different members of these equations do not represent the same values, as they never represent definite values at all.

Also $\circ \times \infty = 0 \div 0$ is wholly indeterminate. We cannot write $a \pm \circ = a$ nor can we write $a \pm \circ = c$, if c is a constant.

(10). By the Binomial Theorem we may demonstrate that, limit

$$\left(1+\frac{1}{x}\right)^{x} = 2.7181, \text{ and } \left(1-\frac{1}{x}\right)^{x} = \frac{1}{2.7181}, \text{ when } x = \infty.$$

But all powers of 1 are 1; $\therefore \circ$ is something real, but different from 0.

Since $1 + \circ$ differs from $1 - \circ$ it is not true that "an infinitesimal must be rejected as having no value in comparison with a finite quantity".

(11). If $a \div 0$ does not properly represent infinity, what interpretation are we to give this expression? We reply that it is the symbol of *impossibility*, not of *quantity*. This will be evident from the discussion of questions which give rise to this symbol. The eq'n $t = a \div (m-n)$ given in (5) is the answer to the problem of the couriers; "In what time will A overtake B?" (m being the rate at which A travels and n that of B.) If $m = n$ or $m - n = 0$, then it is impossible for A to overtake B. To say it will require an infinite time, is to give an affirmative paradoxical form to the negative proposition, A can *never* overtake B.

(12). We have in Trigonometry $\sin x \div \cos x = \tan x$. If $x = \frac{1}{2}\pi$, then $\sin x = 1$, $\cos x = 0$, and we have $1 \div 0 = \tan \frac{1}{2}\pi$. But when $x = \frac{1}{2}\pi$ the secant is parallel to the tangent and hence cannot meet it. Therefore the tangent of $\frac{1}{2}\pi$ is *impossible*, since by its definition it is *terminated* by the secant. The symbol $1 \div 0$ represents this impossibility. The true and correct statement is $\tan \left(\frac{1}{2}\pi \mp \circ\right) = \pm \infty$. That is, as the arc indefinitely approaches $\frac{1}{2}\pi$, the tangent increases without limit; if the arc is less than $\frac{1}{2}\pi$ the tan. is positive, if greater than $\frac{1}{2}\pi$ the tangent is negative; but when the arc $= \frac{1}{2}\pi$ the tangent *vanishes*.

(13). Again, in the problem of the lights,

$$x = \frac{a\sqrt{m}}{\sqrt{m} \pm \sqrt{n}},$$

m and n representing the intensities at a unit's distance and x the distance from the brighter light to the point equally illuminated.

If $m = n$ the first value of x is $\frac{1}{2}a$; the second is $a \div 0$, showing that there is no second point of equal illumination. This is the exact truth.

To say that there is a second point of equal illumination infinitely remote in one direction but none in the other, is to make a false statement. The two intensities being equal, there is as much reason to assume a second point infinitely remote to the right as to the left. But if $a \div 0$ indicates impossibility we have one point only.

Again, if $m > n$ and $a = 0$, then $x = 0$. This case has always presented a difficulty; since there ought to be no point of equal illumination when the intensities are unequal. Now when $x = 0$ and $a = 0$ the expressions for the intensities become

$$\frac{m}{0^2} \quad \text{and} \quad \frac{n}{(0-0)^2},$$

which are symbols of impossibility. The true solution therefore follows the right interpretation of the symbol $a \div 0$; i. e., there is *no* point of equal illumination.

Prof. De Morgan says that he dates his first clear conception of mathematical infinity from the time when he rejected the relation $a \div 0 = \infty$.

(14). When a student I was taught that $a \div 0 = \infty$ by the following reasoning:—Division is a short method of subtraction. To divide a number by 5, we subtract 5 from the number, and then 5 from the remainder and so on until the number is exhausted. The number of subtractions is the quotient. Now to divide 8 by nought we first subtract nought and then subtract nought from the remainder, which is 8, and again subtract nought from each successive remainder and so on for ever, without exhausting the dividend. Therefore the number of subtractions is unlimited and the quotient is infinite.

This reasoning is as falacious as it is specious. To subtract nothing is not subtract any thing. If five books are lying upon the table, how many times can you take away no book? The question is without meaning. To divide by nothing is meaningless if it does not mean not to divide at all. Hence we conclude that $a \div 0$ is not a symbol of value, or symbol of quantity.

(15). Dr. Whewell lays down the following axiom (?) of limits, which has been adopted by Davies and Peck and many American authors:—

"Whatever is true *up to* the limit is true *at* the limit". Let us test this by Trigonometry.

If $x < \frac{1}{2}\pi$, sec x meets tan x. This is true up to the limit $\frac{1}{2}\pi$. Hence

according to the axiom, the secant of $\frac{1}{2}\pi$ meets the tangent which is parallel to it. Again, if $x < \frac{1}{2}\pi$ tan x is positive up to the limit $\frac{1}{2}\pi$; ∴ tan $\frac{1}{2}\pi$ is positive. If $x > \frac{1}{2}\pi$ and decreasing, tan x is negative up to the limit; ∴ tan $\frac{1}{2}\pi$ is negative. Hence also two parallel lines meet in opposite directions and inclose a space; all of which is absurd.

Again, let us test the axiom by analytical geometry. Trace the curve whose equation is

$$y = \sqrt{\frac{x^3}{x-a}}.$$

If x is less than a, y is imaginary. This is true up to the limit, $x = a$; therefore when $x = a$, y is imaginary. Secondly, let x be greater than a, and y is real. This is also true as x decreases up to the limit $x = a$; ∴ when $x = a$ y is real. The results are contradictory, hence the so-called axiom cannot be true.

The true analysis is, if $x < a$, y is *imaginary*, and if $x = a$, y is *impossible;* i. e., the ordinate does not meet the curve. If $x > a$, y is *real* and meets the curve. If $x = a + \circ$, y meets the curve at an indefinitely great distance; hence the line $x = a$, parallel to the axis of y is an asymptote; all of which is rigidly exact.

(16). As a further illustration of absurd conclusions arising from the assumption that $a \div \infty = 0$, let us inscribe a regular polygon of n sides in a given circle. If A, B, C, &c. represent the angular points of the polygon and a, one of the equal angles, the sum of all the angles, $na = 2(n - 2)$. (The right angle being the unit angle.) Hence $a = 2 - (4 \div n)$

Let the number of sides become infinite, then $4 \div n = 0$ and $a = 2$. But if two lines, AB, BC, meet so as to form an angle equal to two right angles, then AB, BC form one straight line. The same is true of BC, CD &c.; hence the entire perimeter is a straight line. Therefore, since the polygon "coincides with the circle", the circumference of a circle is a straight line!

Assumptions from which such conclusions are logically deduced *must be erroneous.*

[We dissent to this conclusion of Prof. Judson. The equation is manifestly true for all finite lines, AB, BC, &c.; but when the number of sides is infinite the lines are reduced to points, which are without length, and therefore can have no curvature : Just as in the motion of a projectile whose initial direction is above the horizon ; we know there is a moment of time in which the projectile neither ascends nor descends but has uniform horizontal motion ; ∴ during that moment its track is not curved, and yet every finite portion of its path is a smooth curve.—Ed.]

(17). Again, the foundation of the Differential Calculus is often laid on the assumption that an infinitesimal, when added to a finite quantity, must be rejected as zero.

The results of (10), which all accept, are a sufficient refutation of this assumption.

Mr. Price in his very valuable Treatise on the Infinitesimal Calculus (Oxford) devotes some ten pages of his introduction to a discussion of the terms infinite and infinitesimal, and the logic of his work is greatly marred by reason of his inexact notion and use of these terms.

Thus, he says that if a grain of aloetic acid be added to five pounds of pure water, it imparts a crimson color to the whole volume. "The grain of acid is divided into thirty-five millions of parts which are so small as to be beyond the limits of our vision; . . . they are *infinitesimal*, though the sum of them is finite; and as they are so small there must be an *infinity* of them." Infinity it would seem depends on the perfection of our organs of vision!

Again, he says, "the distance of the star Capella is 20 billions of miles; but as it is determinable it is finite; *though on the verge of the infinite.* The distance of the stars which have no paralax, he holds, is *infinite.* Here again the infinite is made to depend on the perfection of our instruments of measurement and that of our organs of vision. Surely mathematical science should aim at greater exactness.

Nought, he holds, is a relative term, like small, or great; and one nothing may be infinitely smaller than another. If so, then nought cannot represent the absence of value, as $5 — 5 = 0$. We need another word and another symbol for the *relative* nought.

(18). Messrs. Thomson and Quinby give the following illustration of a false interpretation of $a \div 0 = \infty$. (Algebra, Art. 348, p. 146.) "Given $x^2 + xy = 10$ (1), and $xy + y^2 = 15$ (2), to find x and y.

Let $x = zy$. Then, from (1),

$$y^2 = \frac{10}{z^2 + z} \text{ (5), and from (2) } y^2 = \frac{15}{z + 1} \text{ (6).}$$

From (5) and (6) $10z + 10 = 15z^2 + 15z$ (7); $\therefore z = \frac{2}{3}$ or $—1$.

Substituting $— 1$ for z in (5) or (6) we have $y = \pm \infty$, $\therefore x = \mp \infty$; hence $\infty^2 — \infty^2 = 15$ and $\infty^2 — \infty^2 = 10$".

That these results are incorrect is manifest; for if we eliminate y between (1) and (2) we have an equation of the second degree, which should have two roots only. But if $\pm \infty$ be roots, then an equation of the second deg. may have four roots. Adding (1) and (2), $x^2 + 2xy + y^2 = 25$, $\therefore x + y = \pm 5$. By substituting in (1) $\pm 5x = 10$, or $x = \pm 2$. Substituting in (2), $\pm 5y = 15$, $\therefore y = \pm 3$; and these are the only roots.

The correct interpretation of this example is, since when $z = -1$, $y^2 = 15 \div 0$, $\therefore$ $z = -1$ is an impossible value for (1) and (2.)

Prof. Loomis admits $\pm \infty$ as roots of x and y in simultaneous equations, as in $x^5 + y^5 = a$, $x + y = b$. But the first is divisible by the second, and the resulting eq'n is of the 4th degree. The direct solution (see Young's Algebra) gives 4 roots for x and 4 for y. Can we have 6 roots? If the equations were not simultaneous, we admit that if y increases without limit x will be negative and increase without limit. In simultaneous and independent equations the values of x and y are not indeterminate, and we may by elimination obtain a single equation with one unknown quantity.

Can infinity be a root of an equation of the form $ax^n + bx^{n-1} + \ldots + bx + m = 0$? The theory of equations is utterly at variance with an affirmative answer. We conclude that *infinity is never a root of simultaneous equations.*

(19). Whatever may be the estimate of the value of Dr. Davies' contributions to the educational literature of our country, it must be confessed that his mind never seemed to settle down upon anything as satisfactry, even to himself, in relation to zero and infinity. We search his works in vain for any consistent or rational exposition of this subject. In his Logic of Mathematics, §305, he says, "The terms zero and infinity are employed to designate the *limits* to which decreasing and increasing quantities may be made to approach nearer than any assignable quantity". But (page 302), "The science of mathematics employs no definition which may not be clearly comprehended".

In his view, an infinitesimal is that which has no appreciable value; infinity is that which exceeds our appreciation (Logic of Math., p. 282); and demonstration is that which is free from *"appreciable error"*.

"The common impression", says Dr. Davies, "that mathematics is an exact science founded on axioms too obvious to be disputed and carried forward by a logic too luminous to admit of error, is certainly erroneous in regard to the Infinitesimal Calculus". This frank acknowledgment will be admitted so far as it relates to his own expositions.

It is highly important that teachers require of their pupils greater accuracy of expression, as inaccuracy of language leads to inaccuracy of thought. Thus, the phrases, "*at* infinity", "continued to infinity", "when we reach infinity", and the like, should be wholly discarded   Instead of saying "the tangent is infinite when x equals 90°" we may say the tangent becomes infinite as x approaches 90°. In tracing curves, if $y = x^3 \div (a - x)$, we should say, y becomes infinite as x approaches a; and not "when x equals a".

Again, if $y = ax + b$, and $y = a'x + b'$ be two lines the tangent of their included angle is

$$\tan A = \frac{a - a'}{1 + aa'}.$$

If this is a right angle then $1 + aa' = 0$; for the tangent of a right angle is impossible; not, the tangent is infinite. Again, $\log \circ = -\infty$, not $\log 0 = -\infty$. The log of an infinites'l is negative and increases without limit.

Locke (Book 2, Chapt. 17) clearly discriminates between infinite space and a space infinite. "The idea of infinity", he says, "consists in a suppos'd *endless progression*", . . . "our idea of infinity being, as I think, an *endless growing idea*"; . . . "an endless progression of thought". Again he says "there is nothing more evident than the absurdity of the actual idea of an infinite number".

Is not the theological use of the term, infinite, very closely allied to the mathematical? When it is affirmed that the attributes of the Deity are *infinite*, as his love, wisdom, knowledge, power, is it meant that these may be "indefinitely approached", or simply that they are without limitation, as are duration, space and number?

The writer would be glad to meet with a single example in pure or app'd mathematics in which the view here imperfectly set forth does not afford a rational and consistent solution.

REPLY TO CRITICISM OF EDITOR, P. 108.—The word *limit* is employed in mathematics in two distinct senses — (1) as marking the terminus which cannot be passed; (2) as a constant which may be indefinitely approached.

(1). In the equation of a circle, $x^2 + y^2 = r^2$, the limits of x are $\pm r$. These limits may be reached, as there is no *indefinite approach*.

(2). But if z is a function of x, and x increases or decreases without limit, then z can never reach its limit. "Limit" in Cor. 1. is that defined in (9).

Page (110). Is a curve line "one no part of which is straight", or is it "composed of an infinite number of infinitesimal straight lines"? If the latter then the tang't coincides with the curve for an infinitesimal distance, and we must abandon Euclid and define an infinite number and an infinitesimal distance, as *constants*.

The notion of a line as composed of an infinite number of consecutive p'ts is not mathematically exact. By what process can the sides of a polygon be reduced to points "without length"? Certainly not by increasing the number of the sides without limit. What is meant by the phrase, "when the number of sides is infinite"? And how do we know that the equa ion is not true then? My solution, of course, is that $4 \div n$ is not zero, but an infinitesimal when n increases without limit.

[We admit the difficulty of constructing a line, which has length, of points which are without length; but we cannot perceive that Prof. Judson's treatment of the subject obviates that difficulty.—Ed.]